Portfolio Optimization Under a Minimax Rule

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This paper provides a new portfolio selection rule. The objective is to minimize the maximum individual risk and we use an l_{∞} function as the risk measure. We provide an explicit analytical solution for the model and are thus able to plot the entire efficient frontier. Our selection rule is very conservative. One of the features of the solution is that it does not explicitly involve the covariance of the asset returns.

(Portfolio Selection; Risk Averse Measures; Bicriteria Piecewise Linear Program; Efficient Frontier; Kuhn-Tucker Conditions)

1. Introduction

The portfolio selection problem is of both theoretical and practical interest. Markowitz (1952) laid the foundations for this line of research with his mean-variance (M-V) model. While Markowitz used the portfolio variance as a risk measure, other risk definitions have been proposed. Konno (1990) and Konno and Yamazaki (1991) used the mean absolute deviation as their risk measure. The mean absolute deviation corresponds to an l_1 function, whereas the variance corresponds to an l_2 function. In this paper we propose a more conservative portfolio selection rule whereby the investor minimizes the maximum risk of the individual assets. This risk measure corresponds to an l_{∞} function.

The classic M-V model can be solved analytically for the efficient frontier (see Merton 1972) when short selling is permitted. It has been found that the composition of the optimal portfolio can be very sensitive to estimation errors in the expected returns of the underlying assets; see Chopra and Ziemba (1998), Hensel and Turner (1998), Chopra et al. (1992) and Best and Grauer (1991a, 1991b, 1992). In the case of large-scale optimization problems the relationships between the inputs and the optimal portfolio tends to be obscured (Best and Grauer 1991a). Our model provides a clear connection between the expected returns of the assets and their importance in the optimal portfolio. Under our decision rule there are two steps in the solution. First we rank the individual assets in terms of their expected returns and risks. Second, we compute the optimal properties based on the information contained in the rankings. The ranking rule consists of inequalities among the expected returns. This enables us to see more clearly how the composition of the portfolio varies. There are two important differences between our model and conventional models, such as the M-V model. In our model we do not allow for short selling. We impose this restriction to obtain a simple analytical solution. It is a weakness of the model. In our model correlations among the assets are not taken into account. This is in contrast to the M-V approach where diversification helps to reduce risk. However, we argue that total portfolio risk—under the conventional definition—will be kept small if our risk measure is kept small. In some respects our approach is related to that of Young (1998). In both models there is no short selling and the asset correlations do not enter explicitly into the solutions. The main differences are:

(i) Our model minimizes the expected absolute deviation of the future returns from their mean while Young's model maximises the minimum portfolio return over a set of past returns.

(ii) Young's solution involves linear programming whereas we are able to provide an analytical solution.

2. A New Risk Measure Based on l_{∞}

In this section, we introduce our risk measure and formulate the corresponding portfolio optimization problem with this measure. Assume that an investor has initial wealth M_0 , which is to be invested in n possible assets S_j , j = 1, ..., n. Let R_j be the return rate of the asset S_j , which is a random variable. Let $x_j \ge 0$ be the allocation from M_0 for investment to S_j . (Note that by assuming $x_j \ge 0$ we are concerned with the situation where short selling is not allowed). Thus, the feasible region for the portfolio optimization problem is

$$\mathcal{F} = \left\{ \mathbf{x} = (x_1, \dots, x_n): \sum_{j=1}^n x_j = M_0, x_j \ge 0, \\ j = 1, \dots, n \right\}.$$
 (2.1)

Let E(R) denote the mathematical expectation of a random variable *R*. Define

$$r_j = E(R_j)$$
 and $q_j = E(|R_j - r_j|).$

Namely, r_j and q_j denote the expected return rate of the asset S_j and the expected absolute deviation of R_j from its mean, respectively.

The expected return of a portfolio $\mathbf{x} = (x_1, \dots, x_n)$ is given by

$$r(x_1,\ldots,x_n)=E\left[\sum_{j=1}^n R_j x_j\right]$$

$$= \sum_{j=1}^{n} E(R_j) x_j = \sum_{j=1}^{n} r_j x_j.$$
 (2.2)

The l_{∞} measure we propose is defined as follows. DEFINITION 2.1. The l_{∞} risk function¹ is defined as

$$w_{\infty}(\mathbf{x}) = \max_{1 \le j \le n} E(|R_{j}x_{j} - r_{j}x_{j}|).$$
(2.3)

Let $\mathbf{x} \in \mathcal{F}$. Then $w_{\infty}(\mathbf{x}) = \max_{1 \le j \le n} E(|R_j - E(R_j)|) x_j$ = $\max_{1 \le j \le n} q_j x_j$. This function is explicitly known if the distribution of each random variable R_j is given. For example, if R_j is normally distributed, then it is easy to verify (see Konno and Yamazaki 1991) that

$$w_{\infty}(\mathbf{x}) = \max_{1 \le j \le n} \sqrt{\frac{2}{\pi}} \sigma_j x_j, \qquad (2.4)$$

where σ_i is the standard deviation of R_i . Historical data can also be used to estimate r_i and q_i .

We assume that investors wish to maximize expected return while minimizing their risk level. This is an optimization problem with two criteria in conflict. Under the l_{∞} risk measure as defined above, our portfolio optimization problem can be formulated as a bicriteria piecewise linear program as follows, which is denoted as **POL**_{∞} (the **P**ortfolio **O**ptimization problem with the **l**_{∞} risk measure).

DEFINITION 2.2. The bicriteria portfolio optimization problem \mathbf{POL}_{∞} under the l_{∞} risk measure is formulated as:²

Minimize
$$\left(\max_{1\leq j\leq n}q_jx_j, -\sum_{j=1}^nr_jx_j\right)$$

subject to $x \in \mathcal{F}$,

where a feasible portfolio $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{F}$ is said

¹ See, for example, Başer and Bernhard (1995), for more discussion on the l_x notation and other related minimax measures.

² Minimize (A, B) indicates that A and B are the two criteria to be minimized.

to be **efficient** if there exists no $\mathbf{y} = (y_1, \dots, y_n) \in \mathcal{F}$ such that

$$\max_{1 \le j \le n} q_j y_j \le \max_{1 \le j \le n} q_j x_j, \qquad \sum_{j=1}^n r_j y_j \ge \sum_{j=1}^n r_j x_j,$$

and at least one of the inequalities holds strictly. Accordingly, the function value $(\max_{1 \le j \le n} q_j x_j, -\sum_{j=1}^n r_j x_j)$ is said to be an **efficient point**.

In words, an efficient point is such that there exists no solution better than it with respect to both criteria. The **efficient frontier** is the collection of all efficient points.

By a simple transformation, one can show that POL_{∞} is equivalent to the following Bicriteria Linear Programming (**BLP**) problem

Minimize
$$\left(y, -\sum_{j=1}^{n} r_{j}x_{j}\right)$$

subject to $q_{j}x_{j} \le y$, $j = 1, ..., n$,
 $\mathbf{x} \in \mathcal{F}$.

Now we convert the bicriteria linear programming problem **BLP** into a parametric optimization problem with a single criterion. For a fixed λ , where $0 < \lambda < 1$, the Parametric Optimization problem of **BLP**, denoted as **PO**(λ), is as follows:

Minimize
$$F_{\lambda}(\mathbf{x}, y) = \lambda y + (1 - \lambda) \left(-\sum_{j=1}^{n} r_j x_j \right)$$

subject to $q_j x_j \leq y$, $j = 1, ..., n$,

The equivalence relation between **BLP** and **PO**(λ) is given below (cf. Yu 1974 for proof).

 $\mathbf{x} \in \mathcal{F}$.

PROPOSITION 2.1. Consider the problems **BLP** and **PO**(λ). The pair (**x**, *y*) is an efficient solution of **BLP** if and only if there exists a $\lambda \in (0, 1)$ such that (**x**, *y*) is an optimal solution of **PO**(λ).

One can think of λ as an investor's risk tolerance parameter—the larger the λ , the more risk the investor

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is to tolerate. Because of the equivalence between POL_{∞} and BLP, there exists the same equivalence relationship between POL_{∞} and $PO(\lambda)$. Thus, an optimal solution for $PO(\lambda)$ gives, accordingly, an efficient solution for POL_{∞} . To obtain the efficient frontier of POL_{∞} , one has to know the optimal solutions of $PO(\lambda)$ for all $\lambda \in (0, 1)$.

In §3 and §4 below, we will show that an optimal solution for **PO**(λ) can be derived analytically, and consequently the whole efficient frontier of **POL**_{∞} can also be determined analytically.

3. A Simple Optimal Investment Strategy

Consider the problem **PO**(λ) with a given $\lambda \in (0, 1)$. Note that the parameters $r_j = E(R_j)$ and $q_j = E(|R_j - E(R_j)|)$, j = 1, 2, ..., n, are constants in **PO**(λ). We assume that

$$r_1 \le r_2 \le \dots \le r_n. \tag{3.1}$$

Furthermore, to avoid ambiguity, we assume that there do not exist two assets S_i and S_j , $i \neq j$, such that $r_i = r_j$ and $q_i = q_j$ (if such two assets do exist in the original problem, we may treat them as a single aggregate asset).

3.1. All Assets Are Risky

In this subsection we consider the case where all the assets are risky. We present our main result and provide some discussion of its meaning.

THEOREM 3.1. For any $\lambda \in (0, 1)$, an optimal solution to **PO**(λ) is given by

$$x_{j}^{*} = \begin{cases} \frac{M_{0}}{q_{j}} \left(\sum_{l \in \mathcal{T}^{*}(\lambda)} \frac{1}{q_{l}} \right)^{-1}, & j \in \mathcal{T}^{*}(\lambda), \\ 0, & j \notin \mathcal{T}^{*}(\lambda); \end{cases}$$
(3.2)

$$y^* = M_0 \left(\sum_{l \in \mathcal{T}^*(\lambda)} \frac{1}{q_l} \right)^{-1}, \tag{3.3}$$

where $T^*(\lambda)$ is the set of assets to be invested, which is determined by the following rule:

(a) If there exists an integer $k \in [0, n - 2]$ such that

$$\frac{r_n-r_{n-1}}{q_n} < \frac{\lambda}{1-\lambda}, \qquad (3.4)$$

$$\frac{r_n - r_{n-2}}{q_n} + \frac{r_{n-1} - r_{n-2}}{q_{n-1}} < \frac{\lambda}{1 - \lambda}, \qquad (3.5)$$

$$\frac{r_n - r_{n-k}}{q_n} + \frac{r_{n-1} - r_{n-k}}{q_{n-1}} + \dots + \frac{r_{n-k+1} - r_{n-k}}{q_{n-k+1}} < \frac{\lambda}{1 - \lambda},$$
(3.6)

and

$$\frac{r_n-r_{n-k-1}}{q_n}+\frac{r_{n-1}-r_{n-k-1}}{q_{n-1}}+\cdots+\frac{r_{n-k+1}-r_{n-k-1}}{q_{n-k+1}}$$

$$+\frac{r_{n-k}-r_{n-k-1}}{q_{n-k}} \ge \frac{\lambda}{1-\lambda},$$
(3.7)

then

$$\mathcal{T}^*(\lambda) = \{n, n-1, \ldots, n-k\}.$$
(3.8)

(b) Otherwise, if the condition above is not satisfied by any integer $k \in [0, n - 2]$, then

$$\mathcal{T}^*(\lambda) = \{n, n-1, \dots, 1\}.$$
(3.9)

The proof of this theorem is given in Appendix A. We now discuss the meaning of the results by examining how changes in the portfolio affect its expected return and risk.

Suppose we have a portfolio \mathbf{x}^0 , in which $x_j^0 > 0$ if $j \in \mathcal{T}^0$ and $x_j^0 = 0$ if $j \notin \mathcal{T}^0$. Namely, $\mathcal{T}^0(\lambda)$ is the set of assets to be invested. Assume there is an asset S_h that is not included in $\mathcal{T}^0(\lambda)$. We will now analyse how the performance of the portfolio will change if the allocation x_h to the asset S_h is increased. Let us construct a new portfolio \mathbf{x}' as follows:

$$x'_{j} = \begin{cases} x_{j}^{0} - \Delta_{j}, & \text{if } j \in \mathcal{T}^{0}(\lambda); \\ \Delta_{h}, & \text{if } j = h; \\ 0, & \text{otherwise.} \end{cases}$$
(3.10)

Further, suppose y^0 is the corresponding risk of the portfolio x^0 (that is, (\mathbf{x}^0, y^0) constitutes a solution for the problem **PO**(λ)). We construct a new solution (\mathbf{x}', y') for **PO**(λ) by creating \mathbf{x}' as above and setting

$$y' = y^0 - \Delta_y. \tag{3.11}$$

To meet the constraints of the problem $PO(\lambda)$, we should have

$$\sum_{i=1}^{n} x_{j}' = \sum_{i=1}^{n} x_{j}^{0} - \sum_{j \in \mathcal{T}^{0}(\lambda)} \Delta_{j} + \Delta_{h} = M_{0},$$
$$x_{j}'q_{j} = (x_{j}^{0} - \Delta_{j})q_{j} \leq y^{0} - \Delta_{y}, \quad \text{for } j \in \mathcal{T}^{0}(\lambda),$$
$$x_{h}'q_{h} = \Delta_{h}q_{h} \leq y^{0} - \Delta_{y}.$$

Thus, we can choose Δ_j , Δ_h and Δ_y as positive numbers satisfying

$$\Delta_h = \sum_{j \in \mathcal{T}^0(\lambda)} \Delta_j, \qquad (3.12)$$

$$\Delta_j = \frac{\Delta_y}{q_j}, \qquad j \in \mathcal{T}^0(\lambda), \tag{3.13}$$

$$\Delta_h = (y^0 - \Delta_y) / q_h. \tag{3.14}$$

Recall $F_{\lambda}(\mathbf{x}, y)$ is the objective function of the problem **PO**(λ). We have

$$F_{\lambda}(\mathbf{x}', y') = \lambda(y^{0} - \Delta_{y}) - (1 - \lambda) \sum_{j \in \mathcal{T}^{0}(\lambda)} r_{j}(x_{j}^{0} - \Delta_{j})$$
$$- (1 - \lambda)r_{h}\Delta_{h}$$
$$= F_{\lambda}(\mathbf{x}^{0}, y^{0}) - \lambda\Delta_{y} + (1 - \lambda) \sum_{j \in \mathcal{T}^{0}(\lambda)} r_{j}\Delta_{j}$$
$$- (1 - \lambda)r_{h} \sum_{j \in \mathcal{T}^{0}(\lambda)} \Delta_{j}$$
$$= F_{\lambda}(\mathbf{x}^{0}, y^{0}) - \Delta_{F}, \qquad (3.15)$$

where

$$\Delta_{F} = \lambda \Delta_{y} - (1 - \lambda) \sum_{j \in \mathcal{T}^{0}(\lambda)} r_{j} \Delta_{j} + (1 - \lambda) \sum_{j \in \mathcal{T}^{0}(\lambda)} r_{h} \Delta_{j}.$$
(3.16)

It is easy to see that in the above equation, $\{\lambda \Delta_y\}$ indicates the decrease in the risk, $\{(1 - \lambda) \sum_{j \in \mathcal{T}^0(\lambda)} r_j \Delta_j\}$ is the decrease in the expected return due to the decreases in the allocations to the assets $j \in \mathcal{T}^0(\lambda)$, and $\{(1 - \lambda) \sum_{j \in \mathcal{T}^0(\lambda)} r_h \Delta_j\}$ indicates the increase in the expected return due to the addition of the asset S_h to the set of assets to be invested.

From (3.13), we can rewrite (3.16) as

$$\Delta_F = \left\{ \lambda - (1 - \lambda) \sum_{j \in \mathcal{T}^0(\lambda)} \frac{r_j - r_h}{q_j} \right\} \Delta_y. \quad (3.17)$$

Define $\mathcal{T}_a = \{j: j \in \mathcal{T}^0(\lambda) \text{ and } r_j > r_h\}$ and $\mathcal{T}_b = \{j: j \in \mathcal{T}^0(\lambda) \}$

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 $j \in \mathcal{T}^{0}(\lambda)$ and $r_{j} < r_{h}$. Then, (3.17) can be rewritten as

$$\Delta_F = \Delta_F^a + \Delta_F^b, \tag{3.18}$$

where

$$\Delta_F^a = \left\{ \lambda - (1 - \lambda) \sum_{j \in \mathcal{T}_a} \frac{r_j - r_h}{q_j} \right\} \Delta_y, \qquad (3.19)$$

and

$$\Delta_F^b = \left\{ (1 - \lambda) \sum_{j \in \mathcal{T}_b} \frac{r_h - r_j}{q_j} \right\} \Delta_y.$$
(3.20)

We now have the following few scenarios.

Case 1: $\Delta_F^a > 0$. From (3.19), we can see that $\Delta_F^a > 0$ means that the decrease in the risk value, $\{\lambda \Delta_y\}$, is greater than the decrease in the net expected return. This together with $\Delta_F^b \ge 0$ gives us $\Delta_F > 0$. Thus, it follows from (3.15) that $F_{\lambda}(\mathbf{x}', y') < F_{\lambda}(\mathbf{x}^0, y^0)$. This means that the portfolio \mathbf{x}^0 can still be improved if the asset S_h is included into the set of assets to be invested. This justifies (3.6).

Case 2: $\Delta_F^a \leq 0$ and the set \mathcal{T}_b is empty. In this case, $\Delta_F = \Delta_F^a \leq 0$. This means that increasing the allocation for the asset S_h will either result in a decrease in the net expected return that is greater than the decrease in the risk value (if $\Delta_F^a < 0$), or yield no benefit (if $\Delta_F^a = 0$). Hence, the asset S_h should not be included in the set of assets to be invested. This justifies (3.7).

Case 3: $\Delta_F^a \leq 0$ and the set \mathcal{T}_b is not empty. As \mathcal{T}_b is not empty, there must exist at least one element $m \in \mathcal{T}_b$ such that $r_m < r_h$. We can show, following a similar idea as above (that is, altering the portfolio by removing the asset S_m and then increasing the allocations to other assets accordingly), that in this case the asset S_m should not be included in the set $\mathcal{T}^0(\lambda)$ for investment. This, together with the observation that the asset S_m must meet the condition (3.7) (this is because $r_m < r_h$ and $\Delta_F^a \leq 0$), justifies (3.7).

The above gives a justification of the ranking rule of Theorem 3.1. We now provide several remarks regarding the optimal portfolio as determined by the theorem.

REMARK 3.1. It may be a bit surprising to observe that the investment strategy given by Theorem 3.1

always suggests including those assets with higher return rates first. In other words, an asset with a higher return rate should always be considered before an asset with a lower return rate is considered for selection. The reason for this apparently counterintuitive result is that the actual amount invested in a particular asset also depends on the risk of that asset. Hence, it is possible that the actual investment to an asset with a high return rate is nearly zero, even if it may have been selected according to the rules given by Theorem 3.1. To see this more clearly, consider an example which we assume satisfies $(r_n - r_{n-1})/q_n$ $<\lambda/(1-\lambda)$ and $(r_n - r_{n-2})/q_n + (r_{n-1} - r_{n-2})/q_{n-1}$ $\geq \lambda/(1 - \lambda)$. Thus, it follows from Theorem 3.1 that an optimal investment strategy is to select assets S_n and S_{n-1} only. Further, according to Theorem 3.1, the actual amounts of investment for the assets will be respectively:

$$x_n^* = \frac{M_0}{q_n(1/(q_{n-1}) + 1/q_n)} = \frac{M_0}{q_n/(q_{n-1}) + 1}, \quad \text{and}$$
(3.21)

$$x_{n-1}^{*} = \frac{M_0}{q_{n-1}(1/(q_{n-1}) + 1/q_n)} = \frac{M_0}{1 + (q_{n-1})/q_n}.$$
(3.22)

Clearly, if q_n is much greater than q_{n-1} , then it is possible that x_n^* is nearly zero while x_{n-1}^* is nearly equal to M_0 .

The optimal strategy as described in Theorem 3.1 is a two-phase decision. In the first phase, the assets are selected according to their return rates. Then in the second phase, the actual amounts allocated to those selected assets are determined based on their risks. When the transaction cost for investing an asset is taken into consideration, a very small allocation of funding to the asset may mean that it should in fact be omitted. Thus, under the investment strategy of Theorem 3.1, an asset may be eliminated in either Phase 1 or Phase 2. In Phase 1, it may be eliminated if its return rate is too low, while in Phase 2, it may also be eliminated if its risk is too high.

REMARK 3.2. The optimal investment strategy given by Theorem 3.1 has the property that $x_j^*q_j = y^*$ for all $j \in \mathcal{T}^*(\lambda)$ (and $x_j^* = 0$ for all $j \notin \mathcal{T}^*(\lambda)$). This means that, for the assets selected for investment, we

shall invest them with the amounts such that they have the same risk y^* (note that $E(|R_i x_i - r_i x_i|) = q_i x_i$ represents the risk of investing an amount x_i in asset S_i ; see Definition 2.1). Note that our problem under the l_{∞} measure is to minimize the maximum individual risk, namely, $w_{\infty}(\mathbf{x}) = \max_{1 \le j \le n} E(|R_j x_j - r_j x_j|).$ Theorem 3.1 implies that to achieve this objective, an optimal investment strategy should invest the assets in such a way that their risks are equal. One can see that, if this is not the case, that is, there exists some asset whose risk is less than that of another asset, then the allocation to this asset can be increased. Such a change of allocation will not increase the maximum risk, while the overall expected return will be increased (for details, see the proof of Lemma 4.1 in §4—the part on the inefficiency of the solution (\mathbf{x}^*, y^*) if the risks of the assets selected under (x^*, y^*) are unequal).

REMARK 3.3. Another property of the optimal portfolio given by Theorem 3.1 is that the amounts of x_i^* , j $\in \mathcal{T}^*(\lambda)$, do not depend on the return rates r_i , as long as the set $\mathcal{T}^*(\lambda)$ is selected. This indicates that the expected return rates determine the set of investable assets, but do not influence the magnitudes of the allocations. This property does not seem to exist in portfolio solutions under other models such as the classic M-V model. Why is this a sensible property? The answer is: (i) the information of the expected return rates is exploited already in the selecting rules of Theorem 3.1 when determining the set of investable assets; and (ii) after the assets selected for investment have been determined using the selecting rules, how to minimize the risk of the investment will become the major concern. Under our model, the maximum individual risk is to be minimized. As we discussed above, to achieve this goal, a sensible way is to have all the assets invested carrying the same risk. This leads to the allocations that are independent of the expected return rates.

Remark 3.4. It is easy to see from Theorem 3.1 that a case where we invest all fund M_0 in a single asset is when

$$\frac{r_n - r_{n-1}}{q_n} \ge \frac{\lambda}{1 - \lambda}.$$
(3.25)

Nevertheless, it should be noted that there exist other cases where almost all the fund M_0 should be invested in a single asset. See Remark 3.1 above. A case where all the fund is invested in a single riskless asset will be further addressed in §3.2 below.

REMARK 3.5. Note that the case where we will select all the assets S_j , j = 1, 2, ..., n, for investment is when Condition (3.7) is not satisfied by any $0 \le k \le n - 2$. In this case, $\mathcal{T}^*(\lambda) = \{n, n - 1, ..., 2, 1\}$ and the proportions of the assets in the efficient portfolio are given by (3.2). More specifically, if we assume $M_0 = 1$, from (3.2) we have

$$x_i^* = \frac{1/q_i}{\sum_{k=1}^n 1/q_k}, \quad i = 1, 2, \dots, n.$$
 (3.26)

There exists an interesting relationship³ between this portfolio and the global minimum variance (GMV) portfolio under the M-V model (see, e.g., Haugen 1997). This is analysed as follows.

Suppose the variance of asset S_i is σ_i^2 , i = 1, 2, ..., n, and assume that the assets are uncorrelated. In this case we can show that the proportions of the assets, x_i^{\diamond} , in the GMV portfolio are given below:

$$x_i^{\diamond} = \frac{1/\sigma_i^2}{\sum_{k=1}^n 1/\sigma_k^2}, \quad i = 1, 2, \dots, n.$$
 (3.27)

Moreover, in the situation when the asset returns are normal, we have $q_i = \sqrt{(2/\pi)} \sigma_i$ (see (2.4)), and thus (3.26) can be rewritten as

$$x_i^* = \frac{1/\sigma_i}{\sum_{k=1}^n 1/\sigma_k}, \quad i = 1, 2, \dots, n.$$
 (3.28)

This has a clear analogy with (3.27). Under our model, the efficient portfolio uses $1/\sigma$, while in the M-V model, the GMV portfolio uses $1/\sigma^2$.

REMARK 3.6. In recent years, it has been shown (see Chopra and Ziemba 1998, Hensel and Turner 1998, Chopra et al. 1992, and Best and Grauer 1991a, 1991b, 1992) that the composition of an efficient M-V portfolio can be extremely sensitive to errors in problem inputs. In particular, it has been found that errors in

³ We are very grateful to Professor P. P. Boyle, the department editor for Finance, for pointing out this relationship. The material here has been based on a note kindly provided by him.

the asset means can be much more damaging than errors in other parameters. Therefore, a similar question we may face is how sensitive the solution of our model could be to changes in the asset means. We now discuss this question. Assume that m = n - k - 1 is the asset that satisfies the Condition (3.7). In other words, $\mathcal{T}^*(\lambda) = \{n, n - 1, ..., m + 1\}$ and asset S_m is the first one excluded from the set $\mathcal{T}^*(\lambda)$ (see Theorem 3.1). Let us analyse the following three categories of assets, where δ_i denotes a perturbation of r_i .

(a) On the assets S_j with j < m: All these assets are not selected for investment by Theorem 3.1. Furthermore, this solution remains unchanged as long as $r_j + \delta_j \leq r_m$. This means that the optimal portfolio is unchanged as long as the perturbations are within the following ranges:

$$-\infty \le \delta_j \le r_m - r_j, \quad \text{for } j < m. \tag{3.29}$$

(b) On the asset S_m : Clearly, if (i) $r_m + \delta_m \le r_{m+1}$ and (ii) $\frac{r_n - r_m - \delta_m}{q_n} + \frac{r_{n-1} - r_m - \delta_m}{q_{n-1}} + \cdots$

$$+\frac{r_{m+1}-r_m-\delta_m}{q_{m+1}}\geq\frac{\lambda}{1-\lambda},$$

then the optimal portfolio is unchanged. Let $\zeta_m = \sum_{k=m+1}^{n} (r_k - r_m)/q_k - \lambda/(1 - \lambda)$. Then, to satisfy the above two conditions, it is sufficient for us to have

$$-\infty \leq \delta_m \leq \min\left\{\frac{\zeta_m}{\sum_{k=m+1}^n 1/q_k}, r_{m+1} - r_m\right\}.$$
 (3.30)

(c) On the assets S_j with j > m: Let $\zeta_j = \lambda/(1 - \lambda) - \sum_{k=j+1}^{n} (r_k - r_j)/q_k$, for j > m. Bear in mind that we want to determine an interval for δ_j such that the conditions of Theorem 3.1 remain satisfied. After some development, we can show that a sufficient condition is as follows:

$$\max\left\{\frac{-\zeta_{j}}{\sum_{k=j}^{n} 1/q_{k}}, -\zeta_{m}q_{j}, r_{j-1} - r_{j}\right\} \leq \delta_{j}$$
$$\leq \min\{\zeta_{m+1}q_{j}, r_{j+1} - r_{j}\}, \quad \text{if } j > m+1; \quad (3.31)$$

$$\max\left\{\frac{-\zeta_{j}}{\sum_{k=j}^{n} 1/q_{k}}, -\zeta_{m}q_{j}, r_{j-1} - r_{j}\right\} \le \delta_{j} \le r_{j+1} - r_{j},$$

if $j = m + 1$. (3.32)

In summary, the analysis above indicates that if the

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perturbation δ_i of an asset mean r_i is within the interval of (3.29)–(3.32), then the optimal portfolio under our model will remain unchanged. Note that (3.29)–(3.32) are sufficient conditions, and in many cases these may not be satisfied at all (but the portfolio may still remain unchanged if they are not satisfied). Generally speaking, because the ranking rule for selecting the assets to be invested under our model is given by a set of inequalities (3.4)–(3.7), our model exhibits some robustness against errors in the problem inputs.

On the other hand, we should emphasize that our model can also be quite sensitive in some cases to errors of the problem parameters, such as the means. To illustrate, let us consider an example in which there are three assets with estimated means $r_1 < r_2 < r_3$. Suppose $(r_3 - r_2)/q_3 < \lambda/(1 - \lambda)$ and $(r_3 - r_1)/q_3$ + $(r_2 - r_1)/q_2 \ge \lambda/(1 - \lambda)$. Then, by Theorem 3.1, we choose Assets 2 and 3 and $x_1^* = 0$, $x_2^* = q_3/(q_2)$ $(+ q_3)$, and $x_3^* = q_2/(q_2 + q_3)$. Further, suppose q_2 is much greater than q_3 . Then, $x_2^* \approx 0$, and $x_3^* \approx M_0$. However, there is an error in r_1 and the actual mean r'_1 of Asset 1 satisfies $r_2 < r'_1 < r_3$ and $(r_3 - r'_1)/q_3$ $<\lambda/(1-\lambda)$ and $(r_3 - r_2)/q_3 + (r_1' - r_2)/q_1 \ge \lambda/(1-\lambda)$ $-\lambda$). In this case, we should actually choose Assets 1 and 3 and the portfolio should be: $x_2^{**} = 0$, $x_1^{**} = q_3/(q_1)$ + q_3), and $x_3^{**} = q_1/(q_1 + q_3)$. Further, suppose q_3 is much greater than q_1 . Then, $x_1^{**} \approx M_0$, and $x_3^{**} \approx 0$. In this example, an error in the estimation of r_1 has changed the portfolio almost completely.

3.2. Inclusion of a Riskless Asset

We now consider the case where there exists a riskless asset. Without loss of generality, we may assume that this riskless asset has the lowest return, namely, i = 1 (recall our Assumption (3.1) and note that all risky assets could be eliminated from consideration if their returns are not greater than that of the riskless asset).

Under the assumption above, we have $q_1 = 0$. To generalize the result in §3.1, we first assume that $q_1 = \epsilon > 0$, where ϵ is a sufficiently small number. We then obtain our result by letting $\epsilon \rightarrow 0^+$. Let us now consider the following two cases.

Case 1. According to the rule given in Theorem 3.1 (with $q_1 = \epsilon > 0$), we find that $1 \notin \mathcal{T}^*(\lambda)$. In this case,

it is obvious that the optimal solution for $PO(\lambda)$ as given in Theorem 3.1 is unchanged.

Case 2. According to the rule given in Theorem 3.1 (with $q_1 = \epsilon > 0$), we find that $1 \in \mathcal{T}^*(\lambda)$. In this case, the optimal solution for **PO**(λ) becomes

$$x_{j}^{*} = 0, \quad \forall j \notin \mathcal{T}^{*}(\lambda),$$

$$x_{j}^{*} = \frac{M_{0}}{q_{j}} \left(\frac{1}{\epsilon} + \sum_{l=n-k, l \neq 1}^{n} \frac{1}{q_{l}} \right)^{-1}, \qquad j \in \mathcal{T}^{*}(\lambda),$$

$$y^{*} = M_{0} \left(\frac{1}{\epsilon} + \sum_{l=n-k, l \neq 1}^{n} \frac{1}{q_{l}} \right)^{-1}.$$

Letting $\epsilon \to 0^+$, we obtain $x_j^* = 0$ for all j > 1, $x_1^* = M_0$ and $y^* = 0$.

Clearly, the case where the riskless asset S_1 is selected into the set $\mathcal{T}^*(\lambda)$ for investment happens only when

$$\frac{r_n - r_1}{q_n} + \frac{r_{n-1} - r_1}{q_{n-1}} + \dots + \frac{r_2 - r_1}{q_2} < \frac{\lambda}{1 - \lambda}.$$
 (3.33)

Combining Case 1 with Case 2 above, we have the following result.

THEOREM 3.2. Given any $\lambda \in (0, 1)$. If (3.33) is not satisfied, then the set $\mathcal{T}^*(\lambda)$ of assets to be selected should be determined by (3.4)–(3.8) and the optimal solution should be computed by (3.2) and (3.3). Otherwise, if (3.33) is satisfied, the optimal investment strategy should be to invest all fund M_0 in the riskless asset (where $y^* = 0$).

4. Tracing Out the Efficient Frontier

We now discuss how to determine the efficient frontier of the problem POL_{∞} . Corresponding to the results in §3.1 and §3.2 respectively, let us carry out our analysis in the following two subsections.

4.1. No Riskless Assets Are Involved

First, define:

$$\alpha_{k} = \frac{1}{q_{n}} + \frac{1}{q_{n-1}} + \dots + \frac{1}{q_{n-k+1}},$$

$$k = 1, 2, \dots, n-1,$$
(4.1)

$$\beta_{k} = \frac{r_{n} - r_{n-k}}{q_{n}} + \frac{r_{n-1} - r_{n-k}}{q_{n-1}} + \dots + \frac{r_{n-k+1} - r_{n-k}}{q_{n-k+1}},$$

$$k = 1, 2, \dots, n-1.$$
(4.2)

It is easy to verify that

$$\begin{cases} \beta_{k} = \beta_{k-1} + \alpha_{k}(r_{n-k+1} - r_{n-k}), \\ k = 1, 2, \dots, n-1 \\ \beta_{0} = 0, \end{cases}$$
(4.3)

where α_k can be computed by the following recursive relation:

$$\alpha_k = \alpha_{k-1} + \frac{1}{q_{n-k+1}}, \quad k = 1, 2, \dots, n-1$$

 $\alpha_0 = 0.$
(4.4)

It is clear that (3.4)–(3.7) reduce to determining an integer $k \in [0, n - 2]$ such that:

$$\beta_1 < \frac{\lambda}{1-\lambda}, \quad \dots, \quad \beta_k < \frac{\lambda}{1-\lambda},$$

$$\beta_{k+1} \ge \frac{\lambda}{1-\lambda}.$$
(4.5)

Because $q_j = E(|R_j - E(R_j)|) > 0$ for any j, $\alpha_k > 0$ for k = 1, 2, ..., n - 1 (see (4.4)). Thus, noting that $r_j \le r_{j+1}$ for j = 1, 2, ..., n - 1, we know that

$$\beta_0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{n-1}. \tag{4.6}$$

Therefore, the Conditions (4.5) reduce to:

$$\beta_k < \frac{\lambda}{1-\lambda}, \qquad \beta_{k+1} \ge \frac{\lambda}{1-\lambda}, \qquad (4.7)$$

Or, equivalently,

$$\frac{\beta_k}{1+\beta_k} < \lambda \le \frac{\beta_{k+1}}{1+\beta_{k+1}}.$$
(4.8)

Letting

$$\underline{\lambda}_k = \frac{\beta_k}{1+\beta_k}$$
 and $\overline{\lambda}_k = \frac{\beta_{k+1}}{1+\beta_{k+1}}$, (4.9)

we see that the Conditions (4.5) (or (3.4)–(3.7)) are further equivalent to

$$\lambda \in (\underline{\lambda}_k, \ \overline{\lambda}_k]. \tag{4.10}$$

Bear in mind that we want to determine the efficient

frontier, namely, all the efficient points, corresponding to all possible $\lambda \in (0, 1)$. Recall Theorem 3.1. Given an integer $k \in [0, n - 2]$, it is clear that for all values of λ in an interval $(\underline{\lambda}_k, \overline{\lambda}_k]$, the set $\mathcal{T}^*(\lambda)$ of assets selected for investment remains unchanged, and thus the optimal solution of **PO**(λ) as given by (3.2) and (3.3) remains unchanged. According to the discussions in §2, such a solution corresponds to an efficient point of **POL**_{∞}. Now the question is whether there exist other efficient points for $\lambda \in (\underline{\lambda}_k, \overline{\lambda}_k]$. This is equivalent to asking whether there exist other optimal solutions for **PO**(λ) when $\lambda \in (\underline{\lambda}_k, \overline{\lambda}_k]$.

Our main idea to analyse the efficient frontier consists of the following arguments:

(1) For all $\lambda \in (\underline{\lambda}_k, \overline{\lambda}_k)$, the solution given by (3.2) and (3.3) is the *unique* optimum for **PO**(λ). Thus, there is only one efficient point for **POL**_{∞}.

(2) For $\lambda = \overline{\lambda}_k$, there are *multiple* efficient points for **POL**_{∞}, which, however, can be determined analytically.

The following two lemmas establish these arguments respectively. For notational convenience, we let $\bar{\lambda}_{n-1} = 1$ below.

LEMMA 4.1. For any k = 0, 1, ..., n - 1, if $\lambda \in (\underline{\lambda}_k, \overline{\lambda}_k)$, then the solution given by (3.2) and (3.3) is the unique optimal solution for **PO**(λ).

The proof of this lemma is given in Appendix B.

By substituting (3.2) and (3.3) into the objective functions of **POL**_{∞} (Definition 2.2), one can see that, corresponding to the solution (\mathbf{x}^* , y^*) for $\lambda \in (\underline{\lambda}_k, \overline{\lambda}_k)$, the efficient point of the problem **POL**_{∞} is equal to

$$P_k^* = (y^*, z^*), \tag{4.11}$$

where y^* is given by (3.3), while

$$z^* = -M_0 \left(\sum_{l \in \mathcal{T}^*(\lambda)} \frac{r_j}{q_j} \right) \left(\sum_{l \in \mathcal{T}^*(\lambda)} \frac{1}{q_l} \right)^{-1}.$$
 (4.12)

Using the notation above, we have

LEMMA 4.2. For any k = 0, 1, ..., n - 2, if $\lambda = \overline{\lambda}_k$, then $(y^* - \Delta_{y'} z^* + \Delta_y \overline{\lambda}_k / (1 - \overline{\lambda}_k))$, where

$$0 \le \Delta_{y} \le \frac{y^{*}}{1 + \sum_{j \in \mathcal{I}^{*}(\lambda)} (q_{n-k-1})/q_{j}}, \qquad (4.13)$$

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is an efficient point of the problem POL_{∞} .

The proof of this lemma is given in Appendix C. REMARK 4.1. From Lemma 4.2 and its proof we can see that any solutions which select the assets in $\mathcal{T}^*(\bar{\lambda}_k)$ as well as the asset l = n - k - 1 are optimal to **PO**(λ). The inclusion of an asset which is not in $\mathcal{T}^*(\bar{\lambda}_k)$ decreases the total return, which, however, also reduces the risk. If $\lambda \neq \bar{\lambda}_k$ for $0 \leq k < n - 2$ or $\lambda \in (\bar{\lambda}_{n-2}, 1)$, then such a solution will be dominated by the solution given by (3.2) and (3.3) (Lemma 4.1 implies this fact). However, if $\lambda = \bar{\lambda}_k$ for $0 \leq k \leq n - 2$, the reduction in risk (with a weight equal to $\bar{\lambda}_k$) is just balanced by the reduction in total return (with a weight equal to $1 - \bar{\lambda}_k$), and thus generates an undominated (efficient) point.

In summary, we have the following result on the efficient frontier.

THEOREM 4.1. The efficient frontier of the problem **POL**_{∞} can be determined by considering *n* intervals (λ_k , $\bar{\lambda}_k$), k = 0, 1, ..., n - 1, as well as n - 1 endpoints $\bar{\lambda}_k$, k = 0, 1, ..., n - 2. Specifically, the efficient frontier consists of

(1) the efficient point (y^*, z^*) corresponding to each $(\underline{\lambda}_k, \overline{\lambda}_k)$ with k = 0, 1, ..., n - 1, where y^* and z^* are given by (3.3) and (4.12) respectively; and

(2) the multiple efficient points $(y^* - \Delta_y, z^* + \Delta_y \bar{\lambda}_k/(1 - \bar{\lambda}_k))$ corresponding to each $\bar{\lambda}_k$ with k = 0, $1, \ldots, n - 2$, where Δ_y is governed by (4.13).

4.2. Riskless Assets Are Involved

Similar to §3.2, we assume, without loss of generality, that there is only one riskless asset S_{i_0} ; namely, $q_j > 0$ for $j \neq i_0$ and $q_{i_0} = 0$.

According to Theorem 3.2, the optimal solution for **PO**(λ) is to allocate all fund M_0 to the riskless asset S_{i_0} when (3.22) is satisfied, i.e., $\beta_{n-i_0} < \lambda/(1 - \lambda)$. This is equivalent to $\lambda \in (\lambda_{n-i_0}, 1)$. In this case, it is easy to see that any other solution with an $x_k > 0$ (and thus $x_{i_0} < M_0$), where $k < i_0$, will be worse than the solution with $x_k = 0$ and $x_{i_0}^* = M_0$ (because $r_k \leq r_{i_0}$ but $q_k > q_{i_0} = 0$, any reallocation of the fund M_0 from the asset S_{i_0} to the asset S_k will increase the objective function value of **PO**(λ)). This means that the optimal solution for **PO**(λ) is unique when $\lambda \in (\underline{\lambda}_{n-i_0}, 1)$.

When $\lambda \notin (\underline{\lambda}_{n-i_0}, 1)$, the asset S_{i_0} is not selected by

the rule of Theorem 3.1 and the relevant analysis in §4.2 above is still valid. Therefore, we have

THEOREM 4.2. The efficient frontier of the problem \mathbf{POL}_{∞} can be determined by considering $n - i_0 + 1$ intervals $(\underline{\lambda}_k, \overline{\lambda}_k), k = 0, 1, \dots, n - i_0 - 1$, and $(\underline{\lambda}_{n-i_0}, 1)$, as well as $n - i_0$ endpoints $\overline{\lambda}_k, k = 0, 1, \dots, n - i_0 - 1$. Specifically, the efficient frontier consists of

(1) the efficient point (y^*, z^*) corresponding to each $(\underline{\lambda}_k, \overline{\lambda}_k)$ with $k = 0, 1, ..., n - i_0 - 1$ or $(\underline{\lambda}_k, 1)$ with $k = n - i_0$, where y^* and z^* are given by (3.3) and (4.12) respectively; and

(2) the multiple efficient points $(y^* - \Delta_y, z^* + (\bar{\lambda}_k)/(1 - \bar{\lambda}_k))\Delta_y)$ corresponding to each $\bar{\lambda}_k$ with $k = 0, 1, ..., n - i_0 - 1$, where Δ_y is governed by (4.13).

5. Total Portfolio Risk and Covariances

At first sight, it seems that neither our new risk measure $w_{\infty}(\mathbf{x})$, nor the optimal solution derived, depends on the covariances between assets. Also, it seems that only the risks of the individual assets, rather than the risk of the entire portfolio, are taken care of. This feature of our approach is in marked contrast to the conventional approach which explicitly takes account of the covariances between the assets.

We should point out here that the total portfolio risk is in fact contained in our model, albeit in an implicit way. To be more specific, we will show in the following that the total portfolio risk is bounded above by our risk criterion $w_{\infty}(\mathbf{x})$.

It is well known that the total portfolio risk is usually modeled as a kind of *deviation* of the actual total return from the expected total return. For example, in Markowitz's M-V model, the total portfolio risk is defined to be the variance as follows:

$$w_2(\mathbf{x}) = E\left[\sum_{j=1}^n R_j x_j - \sum_{j=1}^n r_j x_j\right]^2,$$
 (5.1)

which is the expected squared deviation of the actual total return $\sum_{j=1}^{n} R_j x_j$ from the expected total return $\sum_{j=1}^{n} r_j x_j$. Now, instead of considering the expectation of the deviation, let us consider the probability that

this deviation is greater than a prespecified level, namely, $P(|\sum_{j=1}^{n} r_{j}x_{j} - \sum_{j=1}^{n} R_{j}x_{j}| \ge \xi)$, where ξ is a given positive number. Clearly, to make this probability as small as possible is also a way to ensure the deviation of the actual total return from the expected total return as small as possible. This is an alternative measure of the total portfolio risk. By using the Markov inequality (cf. e.g., Leon-Garcia 1994, p. 137), one can obtain

$$P\left(\left|\sum_{j=1}^{n} r_{j}x_{j} - \sum_{j=1}^{n} R_{j}x_{j}\right| \geq \xi\right)$$
$$\leq \frac{1}{\xi} E\left[\left|\sum_{j=1}^{n} R_{j}x_{j} - \sum_{j=1}^{n} r_{j}x_{j}\right|\right]$$
$$\leq \frac{1}{\xi} \sum_{j=1}^{n} E|R_{j} - r_{j}|x_{j} \leq \frac{n}{\xi} w_{\infty}(\mathbf{x}).$$
(5.2)

The above inequality indicates that the total portfolio risk is bounded by $w_{\infty}(\mathbf{x})$ multiplied by a constant n/ξ which is independent of the choice of the portfolios. The total portfolio risk will be small if $w_{\infty}(\mathbf{x})$ is kept small (nevertheless, it may not be true the other way around).

The analysis above shows that the total portfolio risk is compressed by the risk $w_{\infty}(\mathbf{x})$, and what we propose to do is to minimize $w_{\infty}(\mathbf{x})$. Note that the covariances among assets are involved in the total portfolio risk. Nevertheless, we do not have to deal with the covariances directly. The advantage of doing so is obvious even from the implementation point of view: The optimal investment strategy under our model is much easier to compute and implement, and the whole efficient frontier is also much easier to construct.

An interesting question is how the compositions of the portfolios under the l_{∞} model and the M-V model would change, and compare to each other, when the assets to be invested have different degree (or tightness) of correlations. To examine this question, we will now consider a simple example which contains two assets. Let $\sigma_1^2 = E[(R_1 - r_1)^2], \sigma_2^2 = E[(R_2 - r_2)^2],$ and $COV(R_1, R_2) = E[(R_1 - r_1)(R_2 - r_2)]$. Further, let ρ be the correlation coefficient; that is, define ρ

= COV(R_1 , R_2)/($\sigma_1 \sigma_2$). Then, the M-V model can be formulated as follows:

Minimize
$$(w_2(\mathbf{x}), -(r_1x_1 + r_2x_2))$$
 (5.3)

subject to
$$x_1 + x_2 = M_0$$
, (5.4)

$$x_1 \ge 0, \qquad x_2 \ge 0,$$
 (5.5)

where $w_2(\mathbf{x})$ is the variance of the total portfolio (see (5.1) above). In this two-asset problem, one can see that $w_2(\mathbf{x}) = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\rho \sigma_1 \sigma_2 x_1 x_2$. Introducing a parameter τ , where $0 < \tau < 1$, we can convert the above bicriteria problem to a parametric optimization problem as follows:

Minimize
$$\tau(\sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\rho\sigma_1\sigma_2 x_1 x_2)$$

- $(1 - \tau)(r_1 x_1 + r_2 x_2)$ (5.6)

subject to $x_1 + x_2 = M_0$, (5.7)

$$x_1 \ge 0, \qquad x_2 \ge 0.$$
 (5.8)

Denote the optimal solution for the above problem as $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$ ($\hat{\mathbf{x}}$ corresponds to an efficient point for the multicriteria model given by (5.3)–(5.5); cf. Yu 1974). Applying Kuhn-Tucker conditions, after some simplification we can obtain the following results:

(a) Let $A_1 = (\sigma_2^2 - \rho\sigma_1\sigma_2)M_0 + ((1 - \tau)/2\tau)(r_1 - r_2)$ and $A_2 = (\sigma_1^2 - \rho\sigma_1\sigma_2)M_0 + ((1 - \tau)/2\tau)(r_2 - r_1)$. If $A_1 > 0$ and $A_2 > 0$, then

$$\hat{x}_1 = \frac{A_1}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2},$$
(5.9)

$$\hat{x}_2 = \frac{A_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2};$$
 (5.10)

- (b) If $A_1 \le 0$, then $\hat{x}_1 = 0$ and $\hat{x}_2 = M_0$;
- (c) If $A_2 \le 0$, then $\hat{x}_1 = M_0$ and $\hat{x}_2 = 0$.

Now, let us consider the following two cases.

Case 5.1. $\rho = 0$, namely, assume that the two assets are not correlated. Moreover, assume that the parameters in this case have been so chosen that the two assets are all selected under both the l_{∞} and the M-V models. It follows from (5.9) and (5.10) that

$$\hat{x}_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} M_0 + \left(\frac{1 - \tau}{2\tau}\right) \left(\frac{r_1 - r_2}{\sigma_1^2 + \sigma_2^2}\right), \quad (5.11)$$

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$$\hat{\mathbf{x}}_{2} = \frac{\sigma_{1}^{2}}{\sigma_{1}^{2} + \sigma_{2}^{2}} M_{0} + \left(\frac{1 - \tau}{2\tau}\right) \left(\frac{r_{2} - r_{1}}{\sigma_{1}^{2} + \sigma_{2}^{2}}\right).$$
(5.12)

On the other hand, it is easy to see from Theorem 3.1 that

$$x_1^* = \frac{q_2}{q_1 + q_2} M_0, \tag{5.13}$$

$$x_2^* = \frac{q_1}{q_1 + q_2} M_0. \tag{5.14}$$

Note that the role of q_i is similar to that of σ_i^2 . Therefore, comparing (5.11) and (5.12) with (5.13) and (5.14), we can see that the relevant solution \hat{x}_i under the M-V model has an additional term (i.e., the term $((1 - \tau)/2\tau)((r_1 - r_2)/(\sigma_1^2 + \sigma_2^2))$ for x_1 and $((1 - \tau)/2\tau)((r_2 - r_1)/(\sigma_1^2 + \sigma_2^2))$ for x_2). This can be regarded as a compensative term, which makes use of the information given by the return rates r_1 and r_2 to fine-tune the portfolio (\hat{x}_1, \hat{x}_2) .

The effect of the compensation reduces when the difference between r_1 and r_2 decreases. This argument is illustrated by Figures 5.1 and 5.2, where we show the efficient frontiers of the l_{∞} and M-V models,⁴ in the

⁴ In each of the figures, we assumed the two assets follow independent normal distributions, with $\sigma_1^2 = 0.8$, $\sigma_2^2 = 0.4$, and $\rho = 0$. The values of q_1 and q_2 were computed by the relation $q_j = \sqrt{(2/\pi)} \sigma_j$; see (2.4). We altered only the values of r_1 and r_2 in the two figures.



M-V space. The two figures consider respectively the two situations: the difference between r_1 and r_2 is large (Figure 5.1), and small (Figure 5.2). Note that the l_{∞} solution is always dominated by the M-V solution in the M-V space, and therefore the efficient frontier of the l_{∞} model is always below that of the M-V model. Nevertheless, from the two figures we see that the l_{∞} frontier tends to approach the M-V frontier when the difference between r_1 and r_2 decreases.

Case 5.2. $\sigma_1^2 \approx \sigma_2^2 \approx \sigma^2$, namely, assume the variances of the two assets are close to each other. In this case, from (5.9) and (5.10) we have

$$\hat{x}_{1} \approx \frac{\sigma^{2}(1-\rho)M_{0}}{2\sigma^{2}(1-\rho)} + \frac{1-\tau}{2\tau}\frac{r_{1}-r_{2}}{2\sigma^{2}(1-\rho)}$$
$$= \frac{1}{2}M_{0} + \left(\frac{1-\tau}{4\tau\sigma^{2}}\right)\frac{r_{1}-r_{2}}{1-\rho},$$
(5.15)

$$\hat{x}_{2} \approx \frac{\sigma^{2}(1-\rho)M_{0}}{2\sigma^{2}(1-\rho)} + \frac{1-\tau}{2\tau}\frac{r_{2}-r_{1}}{2\sigma^{2}(1-\rho)}$$
$$= \frac{1}{2}M_{0} + \left(\frac{1-\tau}{4\tau\sigma^{2}}\right)\frac{r_{2}-r_{1}}{1-\rho}.$$
(5.16)

Thus, if $\rho \approx 1$, namely, the two assets are highly correlated, the portfolio $\hat{\mathbf{x}}$ under the M-V model can be very sensitive to the parameters. It is possible that a small difference in some parameters will cause $\hat{\mathbf{x}}$ to allocate all the fund M_0 to an asset (e.g., Asset 1) and

nothing to the other asset. The allocation under the l_{∞} model remains as (5.13) and (5.14) (More specifically, $x_1^* \approx x_2^* \approx \frac{1}{2}M_0$, if $q_1 \approx q_2$).

REMARK 5.1. In summary, we have the following observations:

(i) In situations in which the assets have low or no correlations, the allocation to each asset under the M-V model may be further tuned by the information on the return rates. As compared to the portfolio under the l_{∞} model, the portfolio under the M-V model may yield a higher return.

(ii) In situations in which the assets are highly correlated and the variances of the assets are close to each other, the generation of a portfolio under the M-V model can be highly sensitive to the parameters. A small error in the estimation of the parameters may result in a totally different portfolio. Nevertheless, in these situations a diversification in the portfolio of the l_{∞} model is still maintained—this should be a desirable feature to avoid the risk of generating a wrong portfolio due to some small difference in the parameters.

(iii) A special case of (ii) above is when $\tau = 1$. For the M-V model, this corresponds to the global minimum variance portfolio (see also Remark 3.5). In this case, (5.15) and (5.16) reduce to $\hat{x}_1 \approx \hat{x}_2 \approx \frac{1}{2}M_0$, and now the portfolio under our l_{∞} model and that under the M-V model become nearly identical. Since the global minimum variance portfolio is the most conservative solution under the M-V model, this also tends to affirm the argument that our model is a very conservative one.

6. Concluding Remarks

This article addresses the problem of portfolio selection for cautious investors. A portfolio optimization model with a new l_{∞} risk measure has been proposed. A simple scheme has been derived, which generates the efficient portfolio under the l_{∞} model analytically. We have also shown how the whole efficient frontier of the l_{∞} model can be traced out analytically. A simple example is discussed to show the portfolio compositions of the l_{∞} model as compared to the M-V model. The analysis indicates that the l_{∞} model would be more stable when assets to be invested are highly correlated. Nevertheless, the M-V model may generate higher returns when the assets have no or low correlations.

The new l_{∞} model and the related techniques are easy to manipulate and implement in practice. For example, our selection of the efficient portfolio is based on a simple ranking rule, which evaluates one asset at a time, to determine whether or not it should be included into the portfolio. This not only allows a portfolio manager to evaluate which of the current assets are investable, but also enables him to assess the impact of the introduction of any new asset on the efficient portfolio. Also, from the ranking rule, a portfolio manager can see the desirable characteristics of those good assets. Besides, our selection of the optimal portfolio does not involve the correlations among stocks. This releases a portfolio manager from the requirement to relate his portfolio selection to those complicated covariance matrices. The whole efficient frontier of our model can be constructed analytically. This is useful, as it makes it easy to examine the various possible trade-offs between return and risk.

As revealed by recent research in the portfolio selection literature, a common problem with portfolio selection models is their sensitivity to errors in problem parameters, particularly in the means. We have shown that, due to the feature that the ranking rule under our model consists of a set of *inequalities*, our model exhibits some robustness to the errors in the problem inputs. Some conditions under which perturbations in the means would not change the efficient portfolio have been given in Remark 3.6. However, the sensitivity analysis in Remark 3.6 is still very brief and preliminary. Moreover, as we have shown in Remark 3.6, in certain situations our model can also be very sensitive to errors in the means. As our model relies heavily on the means, and there is widespread evidence that the means are very difficult to estimate (see the references cited in Remark 3.6), a more thorough sensitivity analysis on the model is an important topic for further research. Both theoretical analysis and computational evaluation should be helpful.

Our model has the restriction that short selling is not allowed, and all of our results have been obtained with this assumption. While it is well known that the

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removal of this restriction makes the derivation of an optimal portfolio for the M-V model much easier, it is not clear whether this is also true in our model. As short selling also represents an important class of market situations, it is an interesting topic for further research to investigate how the investment strategy of our model would change if short selling were allowed. Furthermore, it would be more interesting to generalize the model to include the constraints that some x_i are subject to upper bounds U_i , some x_i are subject to lower bounds L_i , and some x_i are subject to no restriction.⁵

⁵ We wish to express our sincere gratitude to Professor Phelim Boyle, the department editor for Finance, who provided many valuable suggestions, including a detailed analysis revealing a relationship of our portfolio to the global minimum variance portfolio under the M-V model, and drew our attention to related work in this area. This has led to significant improvements in the paper. The helpful comments and suggestions from the anonymous referees and Professor Robert Heinkel, former department editor, are also much appreciated. Finally, we gratefully acknowledge the support of an RGC Direct Grant (X. Cai); two research grants from the PolyU Research Committee (K.-L. Teo and X. Q. Yang), and a CUHK Mainline Research Grant (X. Y. Zhou).

Appendix A—Proof of Theorem 3.1

We apply the Kuhn-Tucker (K-T) conditions to $PO(\lambda)$. First, let us introduce the Lagrangian of $PO(\lambda)$:

$$L(\mathbf{x}, y, \boldsymbol{\mu}, \lambda_0, \boldsymbol{\gamma}) = \lambda y + (1 - \lambda) \left(-\sum_{j=1}^n r_j x_j \right) + \sum_{j=1}^n \boldsymbol{\mu}_j (q_j x_j - y)$$
$$+ \lambda_0 \left(\sum_{j=1}^n x_j - M_0 \right) - \sum_{j=1}^n \boldsymbol{\gamma}_j x_j.$$

Then, the K-T conditions (see, for example, Zeleny 1981) that an optimal solution (x, y) must satisfy can be written as follows:

$$\frac{\partial L}{\partial y} = \lambda - \sum_{j=1}^{n} \mu_j = 0, \qquad (A.1)$$

$$\frac{\partial L}{\partial x_j} = -(1-\lambda)r_j + \mu_j q_j + \lambda_0 - \gamma_j = 0, \qquad j = 1, \dots, n, \qquad (A.2)$$

$$\sum_{j=1}^{n} x_j = M_{0}, \tag{A.3}$$

$$(q_j x_j - y) \mu_j = 0, \qquad j = 1, \ldots, n,$$
 (A.4)

$$\gamma_j x_j = 0, \qquad j = 1, ..., n,$$
 (A.5)

$$\mu_j \ge 0, \qquad j = 1, \dots, n,$$
 (A.6)

$$\gamma_j \ge 0, \qquad j = 1, \dots, n. \tag{A.7}$$

Define $\mathcal{T}^*(\lambda) = \{j: \mu_j > 0\}$. We let $x_j = 0$, for $j \notin \mathcal{T}^*(\lambda)$. (This is a conjecture, but we shall show in the following that this is in fact correct in terms of satisfying the K-T conditions.) Then, from (A.4) we have $x_i = y/q_i$, if $j \in \mathcal{T}^*(\lambda)$. Thus, from (A.3) we obtain

$$y = M_0 \left(\sum_{l \in \mathcal{T}^*(\lambda)} \frac{1}{q_l} \right)^{-1}.$$
 (A.8)

Therefore,

$$x_{j} = \begin{cases} \frac{M_{0}}{q_{j}} \left(\sum_{l \in \mathcal{T}^{*}(\lambda)} \frac{1}{q_{l}} \right)^{-1}, & j \in \mathcal{T}^{*}(\lambda), \\ 0, & j \notin \mathcal{T}^{*}(\lambda). \end{cases}$$
(A.9)

From (A.5), it follows that if $x_j > 0$, then $\gamma_j = 0$. Thus $\gamma_j = 0$, $\forall j \in \mathcal{T}^*(\lambda)$. For $j \in \mathcal{T}^*(\lambda)$, it is clear from (A.2) that

$$\mu_{j} = \frac{1}{q_{j}} \left[(1 - \lambda)r_{j} - \lambda_{0} + \gamma_{j} \right] = \frac{1}{q_{j}} \left[(1 - \lambda)r_{j} - \lambda_{0} \right].$$
 (A.10)

This together with (A.1) gives us $\lambda = \sum_{l \in \mathcal{I}^*(\lambda)} (1/q_l) [(1 - \lambda)r_l - \lambda_0]$. Therefore,

$$\lambda_0 = \left(\sum_{l \in \mathcal{I}^*(\lambda)} \frac{1}{q_l}\right)^{-1} \left((1-\lambda) \sum_{l \in \mathcal{I}^*(\lambda)} \frac{r_l}{q_l} - \lambda \right).$$
(A.11)

Thus, from (A.10),

$$\mu_{j} = \frac{1}{q_{j}} \left[(1 - \lambda)r_{j} - \left(\sum_{l \in \mathcal{T}^{*}(\lambda)} \frac{1}{q_{l}}\right)^{-1} \times \left((1 - \lambda) \sum_{l \in \mathcal{T}^{*}(\lambda)} \frac{r_{l}}{q_{l}} - \lambda \right) \right], \quad j \in \mathcal{T}^{*}(\lambda), \quad (A.12)$$

and for $j \notin \mathcal{T}^*(\lambda)$,

$$\gamma_j = -(1-\lambda)r_j + \mu_j q_j + \lambda_0 = -(1-\lambda)r_j + \lambda_0.$$
 (A.13)

Clearly, if one can *correctly* determine the set $\mathcal{T}^*(\lambda)$ which ensures that μ_j and γ_j as expressed by (A.12) and (A.13) are all nonnegative, then *y* and *x_j* as given by (A.8) and (A.9), respectively, will be a solution satisfying all the K-T Conditions (A.1)–(A.7).

Our argument is, if there exists an integer $k \in [0, n - 2]$ such that (3.4)–(3.7) hold, then $\mathcal{T}^*(\lambda)$ as given by (3.8) is the set that ensures $\mu_j \ge 0$ and $\gamma_j \ge 0$. The following analysis proves this argument.

By (A.12), it follows that, for any $j \in \mathcal{T}^*(\lambda) = \{n, n - 1, \dots, n - k\}$,

$$\begin{split} \mu_{j} &= \left(\sum_{l \in \mathcal{I}^{*}(\lambda)} \frac{q_{j}}{q_{l}}\right)^{-1} \left[(1-\lambda) \sum_{l \in \mathcal{I}^{*}(\lambda)} \frac{r_{j} - r_{l}}{q_{l}} + \lambda \right] \\ &= \left(\sum_{l \in \mathcal{I}^{*}(\lambda)} \frac{q_{j}}{q_{l}}\right)^{-1} (1-\lambda) \left\{ \left[-\sum_{l=j+1}^{n} \frac{r_{l} - r_{j}}{q_{l}} + \frac{\lambda}{(1-\lambda)} \right] \right. \\ &+ \left. \sum_{l=n-k}^{j} \frac{r_{j} - r_{l}}{q_{l}} \right\} > 0, \quad (by \ (3.6)). \end{split}$$

On the other hand, for j = 1, 2, ..., n - k - 1, from (A.13) and (A.10), we have

$$\begin{split} \gamma_{j} &= -(1-\lambda)r_{j} + \left(\sum_{l\in\mathfrak{T}^{*}(\lambda)}\frac{1}{q_{l}}\right)^{-1} \left((1-\lambda)\sum_{l\in\mathfrak{T}^{*}(\lambda)}\frac{r_{l}}{q_{l}} - \lambda\right) \\ &= \left(\sum_{l\in\mathfrak{T}^{*}(\lambda)}\frac{1}{q_{l}}\right)^{-1} \left((1-\lambda)\sum_{l\in\mathfrak{T}^{*}(\lambda)}\frac{r_{l}-r_{j}}{q_{l}} - \lambda\right) \\ &\geq \left(\sum_{l\in\mathfrak{T}^{*}(\lambda)}\frac{1}{q_{l}}\right)^{-1} \left((1-\lambda)\sum_{l\in\mathfrak{T}^{*}(\lambda)}\frac{r_{l}-r_{n-k-1}}{q_{l}} - \lambda\right) \\ &\geq 0 \quad (by (3.7)). \end{split}$$

The above shows that the K-T Conditions (A.6) and (A.7) are satisfied. This together with the fact that the solution given by (A.8) and (A.9) with the set $\mathcal{T}^*(\lambda)$ of (3.8) also satisfies (A.1)–(A.5) implies that all the K-T conditions are satisfied.

In the case where there does not exist any integer $k \in [0, n - 2]$ such that (3.4)–(3.7) hold, we can show that the solution given by (A.8) and (A.9) with $\mathcal{T}^*(\lambda) = \{n, n - 1, ..., 2, 1\}$, will satisfy all K-T conditions. To do this, we may introduce a dummy asset S_0 with $r_0 = -L$ and $q_0 = L$, where *L* is a sufficiently large positive number. (K-T conditions can be applied even if some parameters, such as an r_j , are negative). Following a similar analysis to the one above, one can show that all the K-T conditions are satisfied.

In summary, because $\mathbf{PO}(\lambda)$ is a convex programming problem, the K-T conditions become necessary and sufficient for optimality and therefore the solution given by (A.8) and (A.9), or equivalently, (3.2) and (3.3), which has been shown to satisfy all the conditions, is optimal. This completes the proof. \Box

Appendix B—Proof of Lemma 4.1

We first consider k = 0, 1, ..., n - 2. Suppose that (\mathbf{x}^0, y^0) , where $\mathbf{x}^0 = (x_1^0, ..., x_n^0)$, is an optimal solution for **PO**(λ). Let $\mathcal{T}^0(\lambda)$ be a set such that $x_j^0 > 0$ if $j \in \mathcal{T}^0(\lambda)$ and $x_j^0 = 0$ if $j \notin \mathcal{T}^0(\lambda)$. We shall first show that, if $\mathcal{T}^0(\lambda) \neq \mathcal{T}^*(\lambda)$ ($\mathcal{T}^*(\lambda)$ is the set determined by Theorem 3.1), then we can find a solution better than (\mathbf{x}^0, y^0) which leads to a contradiction. Note that it will be sufficient for us to consider the following two cases.

Case 1. $\mathcal{T}^{0}(\lambda) \neq \mathcal{T}^{*}(\lambda)$ and there exists at least one *h* such that $h \in \mathcal{T}^{*}(\lambda)$ but $h \notin \mathcal{T}^{0}(\lambda)$, namely, $x_{h}^{0} = 0$.

In this case, by constructing a solution (\mathbf{x}', y') for **PO**(λ) as (3.10) and (3.11), we can show, after some development, that

$$F_{\lambda}(\mathbf{x}', y') = F_{\lambda}(\mathbf{x}^{0}, y^{0}) + \Delta_{y}(1 - \lambda) \left\{ \left(-\frac{\lambda}{1 - \lambda} + \sum_{j > m} \frac{r_{j} - r_{m}}{q_{j}} \right) - \sum_{i \le m} \frac{r_{m} - r_{j}}{q_{j}} \right\}.$$
(B.1)

Because $\sum_{j>m} ((r_j - r_m)/q_j) < \lambda/(1 - \lambda)$ if $m \in \mathcal{T}^*(\lambda)$ (see (3.4)–(3.6)) and $r_m \ge r_j$ if $j \le m$, (B.1) gives us $F_{\lambda}(\mathbf{x}^1, y^1) < F_{\lambda}(\mathbf{x}^0, y^0)$, implying that a feasible solution (\mathbf{x}^1, y^1) exists which is better than the solution (\mathbf{x}^0, y^0) . This contradicts the fact that the solution (\mathbf{x}^0, y^0) is optimal. This means that we must have $\mathcal{T}^0(\lambda) \supseteq \mathcal{T}^*(\lambda)$. We now consider the following case, the result of which will eliminate the possibility that $\mathcal{T}^0(\lambda) \supset \mathcal{T}^*(\lambda)$.

Case 2. $\mathcal{T}^{0}(\lambda) \supset \mathcal{T}^{*}(\lambda)$, namely, there is an *m* with $x_{m}^{0} > 0$, $m \in \mathcal{T}^{0}(\lambda)$ but $m \notin \mathcal{T}^{*}(\lambda)$.

In this case, we can construct a solution (\mathbf{x}'', y'') for **PO**(λ) as follows:

$$x_{j}^{"} = \begin{cases} x_{j}^{0} + \Delta_{j}, & j \in \mathcal{T}^{*}(\lambda), \\ x_{m}^{0} - \Delta_{m}, & j = m, \\ x_{j}^{0}, & \text{otherwise.} \end{cases}$$
$$y_{j}^{"} = y^{0} + \Delta_{u},$$

where Δ_{u} , Δ_{m} , and Δ_{i} are selected to be positive numbers such that

$$\Delta_j = \frac{\Delta_y}{q_j}, \qquad j \in \mathcal{T}^*(\lambda), \tag{B.2}$$

$$\Delta_m = \sum_{j \in \mathcal{T}^*(\lambda)} \Delta_j. \tag{B.3}$$

It is not hard to show that the Relations (B.2) and (B.3) together with the feasibility of (\mathbf{x}^0, y^0) ensure that (\mathbf{x}^2, y^2) is a feasible solution. Similar to (B.1), one can derive

$$F_{\lambda}(\mathbf{x}^{0} + \mathbf{\Delta}_{\mathbf{x}^{\prime}} y^{0} + \mathbf{\Delta}_{y})$$

= $F_{\lambda}(\mathbf{x}^{0}, y^{0}) - \mathbf{\Delta}_{y}(1 - \lambda) \left(-\frac{\lambda}{1 - \lambda} + \sum_{j \in \mathcal{I}^{*}(\lambda)} \frac{r_{j} - r_{m}}{q_{j}} \right).$ (B.4)

Because $\sum_{j \in \mathfrak{I}^*(\lambda)} ((r_j - r_m)/q_j) > \lambda/(1 - \lambda)$ (bearing in mind that $\lambda \in (\underline{\lambda}_m, \overline{\lambda}_m)$), we have $F_{\lambda}(\mathbf{x}^2, y^2) < F_{\lambda}(\mathbf{x}^0, y^0)$, which is again a contradiction.

Combining the results of Cases 1 and 2, we see that $\mathcal{T}^0(\lambda) = \mathcal{T}^*(\lambda)$, namely, any optimal solution (\mathbf{x}^0, y^0) must have the same set of assets selected for investment as that in the solution (\mathbf{x}^*, y^*) . After the set $\mathcal{T}^0(\lambda)$ has been fixed, all assets S_j , $j \in \mathcal{T}^0(\lambda)$, should be invested such that their risks are equal, namely, x_j^0 should be determined such that $q_j x_j^0 = y^0$ for all $j \in \mathcal{T}^0(\lambda)$. If this is not true, that is, there exists an asset S_m in $\mathcal{T}^0(\lambda)$ such that $\delta = y^0 - x_m^0 q_m > 0$, then we can increase the allocation to asset S_m by a positive amount Δ_m to construct a new solution (\mathbf{x}', y') as follows:

$$x'_{j} = \begin{cases} x_{j}^{0} - \Delta_{j}, & j \in \mathcal{T}^{0}(\lambda) \text{ and } j \neq m, \\ x_{j}^{0} + \Delta_{m}, & j = m, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$egin{aligned} \Delta_{j} &= rac{\Delta_{y}}{q_{j}}\,, \qquad orall\,j \in \mathcal{T}^{0}(\lambda) - \{m\}, \ \Delta_{m} &= \sum_{j \in \mathcal{T}^{0}(\lambda) - \{m\}} \Delta_{j}, \quad ext{and} \ \Delta_{y} &= rac{\delta}{q_{m}} \left(\sum_{j \in \mathcal{T}^{0}(\lambda)} rac{1}{q_{j}}
ight)^{-1}. \end{aligned}$$

 $y' = y^0 - \Delta_{y'}$

It is easy to see that such a solution (\mathbf{x}', y') is feasible. Similar to the analysis in Case 1 above, we can show that that $F_{\lambda}(\mathbf{x}', y') < F_{\lambda}(\mathbf{x}^0, y^0)$.

Clearly, if $\mathcal{T}^0(\lambda) = \mathcal{T}^*(\lambda)$, and $q_j x_j^0 = y^0$ for all $j \in \mathcal{T}^0(\lambda)$, then the solution (\mathbf{x}^0, y^0) is exactly the same as (\mathbf{x}^*, y^*) . This proves the uniqueness of (\mathbf{x}^*, y^*) .

What remains to be considered is k = n - 1. According to (3.9), the set $\mathcal{T}^*(\lambda)$ becomes $\{n, n - 1, ..., 2, 1\}$ when $\lambda \in (\underline{\lambda}_{n-1}, 1)$. Thus, Case 2 above is impossible now. As for Case 1, the proof remains the same. In summary, the solution given by (3.2) and (3.3) must be the only optimum for **PO**(λ), if $\lambda \in (\underline{\lambda}_k, \overline{\lambda}_k)$, k = 0, 1, ..., n - 1. This completes the proof. \Box

Appendix C—Proof of Lemma 4.2

Case 1 in the proof for Lemma 4.1 continues to be impossible even if $\lambda = \bar{\lambda}_{k}$, since $0 < \lambda < 1$ and thus (B.1) still implies $F_{\lambda}(\mathbf{x}^{1}, y^{1}) < F_{\lambda}(\mathbf{x}^{0}, y^{0})$. Nevertheless, it is possible to have Case 2, namely, to have an asset S_{l} , where $l \notin \mathcal{T}^{*}(\lambda)$, to be selected in an optimal solution. This is shown below.

Clearly (\mathbf{x}^*, y^*) remains an optimal solution for **PO**(λ) when $\lambda = \overline{\lambda}_k$. Now, we can construct a new solution (\mathbf{x}^0, y^0) from (\mathbf{x}^*, y^*) , by increasing the value of x_i , where l = n - k - 1, to Δ_l and, accordingly, decreasing the values of all x_j by Δ_j for $j \in \mathcal{T}^*(\lambda)$ and y by Δ_y such that

$$q_j(x_j^* - \Delta_j) = y^* - \Delta_y, \qquad j \in \mathcal{T}^*(\lambda), \tag{C.1}$$

$$q_l \Delta_l \le y^* - \Delta_y, \tag{C.2}$$

$$\sum_{j\in\mathcal{I}^*(\lambda)} (x_j^* - \Delta_j) + \Delta_l = M_0.$$
 (C.3)

where Δ_i , Δ_j , and Δ_y are all positive, satisfying:

$$\Delta_j = \frac{\Delta_y}{q_j}, \qquad j \in \mathcal{T}^*(\lambda), \tag{C.4}$$

$$\Delta_y \le y^* - q_l \Delta_l, \tag{C.5}$$

$$\Delta_l = \sum_{j \in \mathcal{I}^*(\lambda)} \Delta_j. \tag{C.6}$$

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Similar to (B.1), we can show that

$$F_{\lambda}(\mathbf{x}^{0} + \boldsymbol{\Delta}_{\mathbf{x}^{\prime}} \ y^{0} + \boldsymbol{\Delta}_{y})$$

= $F_{\lambda}(\mathbf{x}^{0}, \ y^{0}) + \boldsymbol{\Delta}_{y}(1 - \lambda) \left(-\frac{\lambda}{1 - \lambda} + \sum_{j \in \mathcal{I}^{*}(\lambda)} \frac{r_{j} - r_{l}}{q_{j}} \right).$ (C.7)

Because $\sum_{j \in \mathfrak{F}^*(\lambda)} ((r_j - r_l)/q_j) = \lambda/(1 - \lambda)$ when $\lambda = \overline{\lambda}_k$, we have $F_{\lambda}(\mathbf{x}^0 + \mathbf{\Delta}_x, y^0 + \Delta_y) = F_{\lambda}(\mathbf{x}^0, y^0)$. This means that all points $(y^* - \Delta_y, z^* + (\overline{\lambda}_k/(1 - \overline{\lambda}_k))\Delta_y)$ are efficient points for the bicriteria problem **POL**_{*x*}, as long as Δ_y satisfies (C.4)–(C.6). Note that $(y^* - \Delta_y, z^* + (\overline{\lambda}_k/(1 - \overline{\lambda}_k))\Delta_y)$ is the value of the objective function under the solution (\mathbf{x}^0, y^0) , as $y^0 = y^* - \Delta_y$ and

$$z^{0} = -\sum_{j \in \mathcal{F}^{*}(\lambda)} r_{j}(x_{j}^{*} - \Delta_{j}) - r_{l}\Delta_{l}$$

$$= z^{*} + \sum_{j \in \mathcal{F}^{*}(\lambda)} r_{j}\Delta_{j} - \sum_{j \in \mathcal{F}^{*}(\lambda)} r_{l}\Delta_{j}$$

$$= z^{*} + \sum_{j \in \mathcal{F}^{*}(\lambda)} (r_{j} - r_{l}) \frac{\Delta_{y}}{q_{j}}$$

$$= z^{*} + \beta_{k+1}\Delta_{y}$$

$$= z^{*} + \frac{\bar{\lambda}_{k}}{1 - \bar{\lambda}_{k}} \Delta_{y} \quad (\text{cf. (4.2) and (4.9)}).$$

It can be seen that Δ_y within the range of (4.13) satisfies (C.4)–(C.6). This completes the proof. \Box

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Accepted by Phelim P. Boyle; received March 1996. This paper was with the authors 6 months for 6 revisions.